confidence interval (CI)

An interval estimate (→ estimator) that specifies a range of values so that there is a given probability for an unknown parameter to lie within that range. The given probability is known as the level of confidence, c. A confidence interval is specified as

\[ P(a \leq X \leq b) = c \]

i.e. there is a probability of c for the random variable X to take a value between (and including) a and b, where X follows the probability distribution of the unknown parameter. Typical values of c are 0.9, 0.95, 0.99, and 0.999. With increasing levels of confidence, the range of the confidence interval also increases.

Due to the central limit theorem, the random variable \( \bar{X} \), which is the sampling distribution of the mean, is normally distributed (→ normal distribution). So when the unknown parameter is \( \mu \), the population mean, and \( X = \bar{X} \), then X can be normalized (→ standard normal distribution), thereby transforming the above formula to

\[ P(-z \leq Z \leq z) = P\left(\frac{a - \hat{\mu}}{\sigma} \leq Z \leq \frac{b - \hat{\mu}}{\sigma}\right) = c \]

At this point, the quantile function \( F_Z^{-1} \) of Z can be used to find the values for \( \frac{a - \hat{\mu}}{\sigma} \) and \( \frac{b - \hat{\mu}}{\sigma} \) and hence for a and b:

\[ z = F_Z^{-1}(c + \frac{\alpha}{2}) \]

Because the normal distribution is a two-tailed distribution, half the size of the critical region has to be added to the confidence level c (where \( \alpha = 1 - c \)). Once z is known, the confidence interval can be specified as

\[ P(-z\sigma + \hat{\mu} \leq X \leq z\sigma + \hat{\mu}) = c \]

The meaning of this statement is that there is a \( \alpha = 1 - c \) probability for the parameter \( \mu \) to lie outside of the given interval while the sample X was skewed by chance. See also: hypothesis test, significance, level of significance.
Figure **CFI**: confidence interval on the standard normal distribution with confidence level $c = 0.95$; light gray area: (probability of the) confidence interval; dark gray area: (probability of the) critical regions; significance level $\alpha = 0.05$, two critical regions with probability $\frac{\alpha}{2} = 0.025$.

Example: Given a sampling distribution of the mean $\bar{X}$ of diameters of marbles with estimated mean $\hat{\mu} = 12.5\,mm$ and $\sigma = 0.2\,mm$, the confidence interval for the mean at a $c = 0.95$ level of confidence would be computed as follows:

$$P(-z \leq Z \leq z) = P\left(\frac{a - 12.5}{0.2} \leq Z \leq \frac{b - 12.5}{0.2}\right) = 0.95$$

$$z = F_Z^{-1}(0.95 + \frac{0.05}{2}) \approx 1.96$$

$$P(-1.96 \cdot 0.2 + 12.5 \leq X \leq 1.96 \cdot 0.2 + 12.5) \approx 0.95$$

$$P(12.11 \leq \bar{X} \leq 12.89) \approx 0.95$$

I.e., the probability for the true mean $\mu$ of the diameter to not lie within the given interval (and the sample being skewed by chance) would be $1 - 0.95 = 0.05$. See figure CFI for an illustration.